CALCULS 2 - Chapter 8 Review

① $a_n = \frac{2+n^3}{1+2n^3}$ CONV, $\lim_{n\to\infty} a_n = \frac{1}{2}$

② $a_n = \frac{9^{n+1}}{10^n} = \frac{9^n \cdot 9}{10^n} = \left(\frac{9}{10}\right)^n \cdot 9$ CONV $\lim_{n\to\infty} a_n = 0$

③ $a_n = \frac{n^3}{1+n^2}$ DIV, $\lim_{n\to\infty} a_n = \infty$

④ $a_n = \frac{n}{\ln n}$ $\lim_{n\to\infty} = \frac{\infty}{\infty} \to \lim_{n\to\infty} \frac{1}{1/n} = \infty$ DIV

⑤ $a_n = \sin n$ Bounded, but DIV. ($\lim_{n\to\infty} a_n$ DNE)

⑥ $a_n = \frac{\sin n}{n}$, $\lim_{n\to\infty} a_n = 0$ CONV

⑦ $a_n = \left(1+\frac{3}{n}\right)^{4n}$ $\lim_{n\to\infty} = 1^\infty$, Let $y = \left(1+\frac{3}{x}\right)^{4x}$ Then $\ln y = 4x \ln\left(1+\frac{3}{x}\right) = \frac{4\ln\left(\frac{x+3}{x}\right)}{x^{-1}} = f(x)$

$\lim_{x\to\infty} f(x) = \frac{0}{0} \to \lim_{x\to\infty} \left[\frac{4\left(\frac{x}{x+3}\right)\left(-\frac{3}{x^2}\right)}{-\frac{1}{x^2}}\right] = \lim_{x\to\infty} \left[\frac{12x}{x+3}\right] = 12$ $\lim_{x\to\infty} \ln y = 12$

$\lim_{x\to\infty} y = e^{12}$, CONV

⑩ $\sum_{n=1}^{\infty} \frac{n^2+1}{n^3+1}$ Choose $b_n = \frac{n^2}{n^3} = \frac{1}{n}$ (DIV p-series), $\frac{a_n}{b_n} = \frac{n^3+n}{n^3+1}$ $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = 1 \Rightarrow \sum_{n=0}^{\infty} a_n$ DIV LIMIT COMPARISON

⑪ $\sum_{n=1}^{\infty} \frac{n^3}{5^n}$ $\frac{a_{n+1}}{a_n} = \frac{(n+1)^3}{5^{n+1}} \cdot \frac{5^n}{n^3} = \frac{1}{5}\left(\frac{n+1}{n}\right)^3$, $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{5} \Rightarrow \sum a_n$ CONV RATIO

⑫ $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$ $b_{n+1} < b_n$ and $\lim_{n\to\infty} b_n = 0 \Rightarrow \sum a_n$ CONV ALT. SERIES

⑭ $\sum_{n=1}^{\infty} \ln\left(\frac{n}{3n+1}\right)$ $\lim_{n\to\infty} a_n = \ln\left(\frac{1}{3}\right) \neq 0 \Rightarrow \sum a_n$ DIV DIV. TEST

⑮ $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\sqrt{n}}{n+1}$, $b_{n+1} < b_n$ and $\lim_{n\to\infty} b_n = 0 \Rightarrow \sum a_n$ CONV. ALT SERIES

⑯ $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$, $\int_2^{\infty} \frac{1}{x(\ln x)^2}\,dx$ let $u = \ln x$, $du = \frac{1}{x}\,dx$ $\int_{\ln 2}^{\infty} \frac{1}{u^2}\,du = -\frac{1}{u}\Big]_{\ln 2}^{\infty} = 0 + \frac{1}{\ln 2} \Rightarrow \sum a_n$ CONV. INTEGRAL TEST

⑰ $\sum_{n=1}^{\infty} \frac{1\cdot3\cdot5\cdots(2n-1)}{5^n\, n!}$, $\frac{a_{n+1}}{a_n} = \frac{1\cdot3\cdot5\cdots(2n-1)(2n+1)}{5^{n+1}(n+1)!} \cdot \frac{5^n\, n!}{1\cdot3\cdot5\cdots(2n-1)} = \frac{2n+1}{5(n+1)}$ $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{2}{5} \Rightarrow \sum a_n$ CONV RATIO TEST

⑱ $\sum_{n=1}^{\infty} \frac{(-5)^{2n}}{n^2 9^n}$, $\frac{a_{n+1}}{a_n} = \frac{(-5)^{2n+2}}{(n+1)^2 9^{n+1}} \cdot \frac{n^2 9^n}{(-5)^{2n}} = \frac{(-5)^2}{9}\left(\frac{n}{n+1}\right)^2$, $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{25}{9} \Rightarrow \sum a_n$ DIV RATIO TEST

⑲ $\sum_{n=1}^{\infty} \frac{2^{2n+1}}{5^n} = \frac{8}{5} + \frac{32}{25} + \frac{128}{125} + \cdots$ Geometric $\left(r = \frac{4}{5}\right)$ $\sum_{n=1}^{\infty} a_n = S_\infty = \frac{a_1}{1-r} = \frac{8/5}{1-4/5} = \boxed{8}$

⑳ $\sum_{n=1}^{\infty} \frac{1}{n(n+3)}$, $f(x) = \frac{1}{x(x+3)} = \frac{A}{x} + \frac{B}{x+3} \Rightarrow 1 = A(x+3) + Bx$

Choose $x = 0 \Rightarrow A = 1/3$

$x = -3 \Rightarrow B = -1/3$

$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1/3}{x} - \sum_{n=1}^{\infty} \frac{1/3}{x+3} = \left[\frac{1}{3} + \frac{1}{6} + \frac{1}{9} + \frac{1}{12} + \frac{1}{15} + \cdots\right] - \left[\frac{1}{12} + \frac{1}{15} + \frac{1}{18} + \cdots\right] = \frac{1}{3} + \frac{1}{6} + \frac{1}{9}$

$= \frac{6+3+2}{18} = \boxed{\frac{11}{18}}$

㉓ $1.2345345345\cdots$

$= 1.2 + [.0345 + .0000345 + .0000000345 + \cdots]$

$= 1.2 + \frac{.0345}{1 - .001} = 1.2 + \frac{.0345}{.999} = \frac{6}{5} + \frac{23}{666} = \frac{3996 + 115}{3330} = \boxed{\frac{4111}{3330}}$

(24) $\sum_{n=1}^{\infty} (\ln x)^n = (\ln x)^1 + (\ln x)^2 + (\ln x)^3 + \cdots$ Geometric $r = \ln x$ $\Rightarrow$ Converges when $|\ln x| < 1$ $\Rightarrow$ $\boxed{e^{-1} < x < e}$

causes $\sum a_n$ to CONVERGE

(25) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^5}$ to 4 decimals $\quad b_7 = \frac{1}{7^5} = \frac{1}{16807} \approx .0000595$

$$S_6 = 1 - \frac{1}{32} + \frac{1}{243} - \frac{1}{1024} + \frac{1}{3125} - \frac{1}{7776} = .97208 \text{ or } \boxed{.9721}$$

(26) $\sum_{n=1}^{\infty} \frac{1}{n^6}$ Find S_5 $\quad \sum_{n=1}^{5} \frac{1}{n^6} = 1 + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \frac{1}{5^6} \approx 1.017304883$

Estimate R_5 with "Remainder Estimate for Integral Test": $\int_6^{\infty} \frac{1}{x^6}\,dx \le R_5 \le \int_5^{\infty} \frac{1}{x^6}\,dx$

$$-\frac{1}{5x^5}\Big]_6^{\infty} \le R_5 \le -\frac{1}{5x^5}\Big]_5^{\infty} \Rightarrow \frac{1}{5\cdot 6^5} \le R_5 \le \frac{1}{5\cdot 5^5} = .000064 \ \left(\text{Maximum Error}\right)$$

b) For sum to be accurate to 5 DECIMALS, S_9 is needed $\sum_{n=1}^{9} \frac{1}{n^5} \approx \boxed{1.01734}$

(27) $\sum_{n=1}^{8} (2+5^n)^{-1} = \frac{1}{2+5} + \frac{1}{2+5^2} + \frac{1}{2+5^3} + \frac{1}{2+5^4} + \frac{1}{2+5^5} + \frac{1}{2+5^6} + \frac{1}{2+5^7} + \frac{1}{2+5^8} \approx .18976224$

Accurate to at least 6 Decimals

Note that $a_n < b_n = \frac{1}{5^n}$ $\quad$ Using $\sum_{n=1}^{\infty} b_n$, $R_8 = \sum_{n=9}^{\infty} \frac{1}{5^n} = \frac{1/5^9}{1 - 1/5} = \frac{1}{4\cdot 5^8}$

Therefore for $\sum_{n=1}^{\infty} a_n$, $R_8 < \frac{1}{4\cdot 5^8} = 6.4 \times 10^{-7}$

(31) $\sum_{n=1}^{\infty} \frac{(x+2)^n}{n\,4^n}$, $\quad \frac{a_{n+1}}{a_n} = \frac{(x+2)^{n+1}}{(n+1)4^{n+1}} \cdot \frac{n \cdot 4^n}{(x+2)^n} = \frac{x+2}{4}\left(\frac{n}{n+1}\right) \quad \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x+2}{4}\right|$

Therefore $\sum_{n=1}^{\infty} a_n$ converges when $-1 < \frac{x+2}{4} < 1 \Rightarrow -6 < x < 2$

Test $x = -6$: $\sum \frac{(-4)^n}{n\,4^n}$ Converges by ALT. SERIES TEST

Test $x = 2$: $\frac{4^n}{n\,4^n}$ Divergent p-series

Therefore $I = [-6, 2)$ + $R = 4$

(33) $\sum_{n=0}^{\infty} \frac{2^n (x-3)^n}{\sqrt{n+3}}$, $\quad \frac{a_{n+1}}{a_n} = \frac{2^{n+1}(x-3)^{n+1}}{\sqrt{n+4}} \cdot \frac{\sqrt{n+3}}{2^n (x-3)^n} = 2\left(\frac{n+3}{n+4}\right)^{1/2}(x-3) \quad \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |2(x-3)|$

Therefore $\sum_{n=1}^{\infty} a_n$ converges when $-1 < 2(x-3) < 1 \Rightarrow \frac{5}{2} < x < \frac{7}{2}$

Test $x = 5/2$: $\sum \frac{(-1)^n}{\sqrt{n+3}}$ CONVERGES (ALT. SERIES TEST)

Test $x = \frac{7}{2}$: $\sum_{n=0}^{\infty} \frac{1}{\sqrt{n+3}} = \sum_{n=3}^{\infty} \frac{1}{\sqrt{n}}$ DIVERGES $(p = 1/2)$ Therefore $I = [\frac{5}{2}, \frac{7}{2})$ $R = \frac{1}{2}$

(36) $f(x) = \cos x$, $a = \pi/3$ FIND TAYLOR SERIES

$f' = -\sin x$ $\quad f'(a) = -\sqrt{3}/2$

$f'' = -\cos x$ $\quad f''(a) = -1/2$

$f''' = \sin x$ $\quad f'''(a) = \sqrt{3}/2$

$f^{(4)} = \cos x = f(x)$ $\quad f^{(4)}(a) = 1/2$

$$f(x) = \frac{1}{2} + \frac{-\sqrt{3}/2}{1!}(x - \pi/3) + \frac{-1/2}{2!}(x - \pi/3)^2 + \frac{\sqrt{3}/2}{3!}(x - \pi/3)^3 + \frac{1/2}{4!}(x - \pi/3)^4 + \cdots$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{2(2n)!}\left(x - \frac{\pi}{3}\right)^{2n} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}\sqrt{3}}{2(2n+1)!}\left(x - \frac{\pi}{3}\right)^{2n+1}}$$

Ch 8 Review- continued

(37) $f(x) = \frac{x^2}{1+x} = x^2\left[\frac{1}{1+x}\right] = x^2 \sum_{n=0}^{\infty} (-1)^n x^n = \boxed{\sum_{n=0}^{\infty} (-1)^n x^{n+2} \qquad R=1}$

(38) $f(x) = \tan^{-1}(x^2) = \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n+1}}{2n+1} = \boxed{\sum_{n=0}^{\infty} \frac{x^{4n+2}}{2n+1} \qquad R=1}$

(39) $f(x) = \ln(1-x) \qquad f'(x) = \frac{-1}{1-x} = -\sum x^n = -1 - x - x^2 - x^3 - \cdots$

$f(x) = \int f'(x)\,dx = C - x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \cdots, \; f(0)=0 \Rightarrow C=0, \; f(x) = \boxed{-\sum_{n=1}^{\infty} \frac{1}{n}x^n \qquad R=1}$

(43) $f(x) = \frac{1}{\sqrt[4]{16-x}} = 16^{-1/4}\left[1 - \frac{x}{16}\right]^{-1/4}$, Use Binomial Series: $f(x) = \frac{1}{2}\sum_{n=0}^{\infty} \binom{-1/4}{n}\left(-\frac{x}{16}\right)^n$

$f(x) = \frac{1}{2}\left[1 + \frac{-1/4}{1!}\left(-\frac{x}{16}\right)^1 + \frac{-1/4(-5/4)}{2!}\left(-\frac{x}{16}\right)^2 + \frac{-1/4(-5/4)(-9/4)}{3!}\left(-\frac{x}{16}\right)^3 + \cdots\right.$

$f(x) = \frac{1}{2} + \frac{1}{2}\sum_{n=1}^{\infty} \frac{1\cdot5\cdot9\cdots(4n-3)}{n!\,4^n\cdot16^n}x^n = \boxed{\frac{1}{2} + \sum_{n=1}^{\infty} \frac{1\cdot5\cdot9\cdots(4n-3)}{n!\,2^{6n+1}}x^n}$

$\left|-\frac{x}{16}\right| < 1$

$|x| < 16$

$R=16$

(45) $\int \frac{e^x}{x} = \frac{1}{x}\int\left[1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right]dx = \int\left[\frac{1}{x} + 1 + \frac{x}{2!} + \frac{x^2}{3!} + \cdots\right]dx$

$= C + \ln|x| + x + \frac{x^2}{2\cdot2!} + \frac{x^3}{3\cdot3!} + \cdots = \boxed{C + \ln|x| + \sum_{n=1}^{\infty} \frac{x^n}{n\cdot n!}}$

(46) $\int_0^1 \sqrt{1+x^4}\,dx$, Use Binomial Series: $\int_0^1 \left[1 + \frac{1/2}{1!}x^4 + \frac{1/2(-1/2)}{2!}x^8 + \frac{1/2(-1/2)(-3/2)}{3!}x^{12} + \cdots\right]dx$

$= x + \frac{1}{10}x^5 - \frac{1}{72}x^9 + \frac{1}{208}x^{13} - \cdots\Big]_0^1 = 1 + \frac{1}{10} - \frac{1}{72} + \frac{1}{208} - \cdots \qquad b_3 = \frac{1}{208} \Rightarrow R_2 < .01$

$\int_0^1 \sqrt{1+x^4}\,dx \approx S_2 = 1 + \frac{1}{10} - \frac{1}{72} = 1.086\overline{1} \qquad \int_0^1 \approx \boxed{1.09}$

(47) $f(x) = \sqrt{x} \qquad a=1 \qquad n=3 \qquad 0.9 \le x \le 1.1$

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$x^{1/2}$	1
1	$\frac{1}{2}x^{-1/2}$	$\frac{1}{2}$
2	$-\frac{1}{4}x^{-3/2}$	$-1/4$
3	$\frac{3}{8}x^{-5/2}$	$3/8$

$\sqrt{x} \approx T_3 = 1 + \frac{1/2}{1!}(x-1) + \frac{-1/4}{2!}(x-1)^2 + \frac{3/8}{3!}(x-1)^3$

$\boxed{T_3 = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3}$